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Un Método de Control $\mathcal{H}_2 - \mathcal{H}_\infty$ para la Admisibilidad Robusta de Sistemas Descriptores LPV a Tiempo Discreto

A $\mathcal{H}_2 - \mathcal{H}_\infty$ Control Setting for Robust Admissibilization of Descriptor LPV Systems to Discrete-Time

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Resumen

La contribución de este trabajo está centrada en la admisibilidad robusta de sistemas descriptores (DS) lineales con a tiempo discreto y con parámetros variantes (LPV). La técnica está basada en la síntesis de controladores robustos que permiten garantizar índices de desempeño en $\mathcal{H}_2 - \mathcal{H}_\infty$ *para el sistema en 1lazo cerrado. Los DS se consideran que tienen incertidumbres de tipo politópicas y perturbaciones externas, tal como se describe por:*

$$E(\rho)x_{k+1} = A(\rho)x_k+B_\omega(\rho)\omega_k+B(\rho)u_k;\ y_k = C(\rho)x_k+D(\rho)\omega_k$$

donde ρ es la variación paramétrica. De la caracterización de las normas $\mathcal{H}_2 = \mathcal{H}_\infty$ *como desigualdades matriciales lineales (LMI), se presentan las condiciones necesarias y suficientes para la admisibilidad en forma de LMI estricta, lo cual permite diseñar los controladores robustos por medio de realimentación de estados o por realimentación estática de la salida (SOF) por medio de herramientas computacionales.*

Palabras clave: Sistemas Descriptores, Sistemas LPV, Admisibilidad Robusta, Normas H_2-H_∞.

Abstract

The contribution of this work is centered in the robust admissibilization of discrete-time linear descrip- tor systems (DS) with variant parameters (LPV). The technique is based on the synthesis of robust con- trollers that allow to obtain performance index in $\mathcal{H}_2 - \mathcal{H}_\infty$ *for the closed-loop system. The DS are considered to have parametric uncertainties of polytopic type and external perturbations, as described by:*

$$E(\rho)x_{k+1} = A(\rho)x_k+B_\omega(\rho)\omega_k+B(\rho)u_k;\ y_k = C(\rho)x_k+D(\rho)\omega_k$$

where ρ is the parametric variation. From the characterization of the $\mathcal{H}_2 = \mathcal{H}_\infty$ *norms as a Linear Matrix Inequality (LMI), the necessary and sufficient admissibility conditions in strict LMI form are derived, which allow to design the robust controllers by state feedback or static output feedback (SOF) using computational tools.*

Keywords: Descriptor Systems, LPV Systems, Robust Admissibility, H_2-H_∞ norms.

1 Introducción

Since its introduction in 1977 (Luenberger, 1977), descriptor systems (DS) have been one of the main research fields within control theory. Unlike their regular counterparts in state space, a DS allows a representation that incorporates algebraic constraints in their physical variables.

On the other hand, the context of linear parameter variable (LPV) systems refers to linear dynamical systems whose state-space representations depend on exogenous non-stationary parameters (Shamma, 2012). LPV systems are a generalization of LTV systems (Duan, Yu, 2013; Briat, 2008).

When there are combined the modeling of physical systems with uncertain parameters, there arise dynamic systems representing uncertain DS. As is well known, for modeling many applications and technical processes, only approximate models are available, so that the analysis of DS subject to uncertainties has been a very active research line, (Chadli et al., 2017; Feng, Yagoubi, 2017; Zhang et al., 2019).

In that order of ideas, one of the main research topics has been the robust admissibility of DS, which has been focused on the state feedback control and $\mathcal{H}_\infty$-norm,(González et al., 2017; Barbosa et al., 2018; Rodríguez et al., 2018; Chang, Wang, 2021). In order to extend these results to the case of $\mathcal{H}_2$ control, in this paper the robust admissibilization of discrete-time linear descriptor systems (DS) with variant parameters (LPV) is studied. A technique for robust admissibility is presented, which consists of the design of robust controllers based on the characterization of the $\mathcal{H}_2$- norm as LMIs, (Pipeleersa et al., 2009; Hilhorst et al., 2014).

Thus, in this paper the robust admissibilization of DS type LPV is studied. A technique for robust admissibility is presented, which consists of the design of robust controllers based on the characterization of the $\mathcal{H}_2$- norm as LMIs, i.e.

Let be discrete-time linear system

$$x_{k+1} = Ax_k \quad (1)$$

then, the following results provide LMI formulation of the robust stability condition:

Lemma 1 (Robust stability). 1) *There exists a matrix* $P = P^T > 0$ *such that*

$$A^TPA - P < 0 \quad (2)$$

2) *There exist a matrix* $P = P^T > 0$ *and a matrix* G *such that*

$$\begin{bmatrix} -P & A^TG^T \\ GA & -G - G^T + P \end{bmatrix} < 0 \quad (3)$$

Proof. See (Grman et al. 2005; Scholz 2015).

If (1) is a polytopic LPV system, i.e.

$$x_{k+1} = A(\rho)x_k = \sum_{i=1}^{N} \rho_i A_i x_k, \quad \text{with} \sum_{i=1}^{N} \rho_i = 1, \forall \rho_i \geq 0 \quad (4)$$

then:

Lemma 1.2 (Robust stability for LPV system). *There exist a matrix* $P_i = P_i^T > 0$ *and a matrix* G *such that*

$$\begin{bmatrix} -P_i & A_i^TG^T \\ GA_i & -G - G^T + P_i \end{bmatrix} < 0, i = 1,2,\dots,N \quad (5)$$

Proof. See (Grman et al. 2005; G. Zhang, Xia, and Shi 2008).

Consider the discrete LTI system defined by

$$\begin{aligned} x_{k+1} &= Ax_k + B\omega_k \\ z_k &= Cx_k + D\omega_k \end{aligned} \quad (6)$$

where $x \in \mathfrak{R}^n$ are the states, $\omega_k \in \mathfrak{R}^m$ are exogenous inputs (noise, disturbance); and $z_k \in \mathfrak{R}^q$ are controlled outputs. The matrices A, B, C and D will be of appropriate dimensions.

Lemma 1.3 (Relaxed $\mathcal{H}_2$ performance). *Consider the system* (6). *For* $P = P^T > 0$, *the following statements are equivalent:*

i) *A is stable and* $\| C(z\mathbb{I} - A)^{-1}B \|_2 < \mu$.

ii) *There exist* $P = P^T \in \mathfrak{R}^{n\times n}$, $W = W^T \in \mathfrak{R}^{q\times q}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} P & PA & PB \\ A^TP & P & 0 \\ B^TP & 0 & \mathbb{I} \end{bmatrix} > 0, \begin{bmatrix} W & C & D \\ C^T & P & 0 \\ D^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad (7)$$

iii) *There exist* $P = P^T \in \mathfrak{R}^{n\times n}$, $W = W^T \in \mathfrak{R}^{q\times q}$, $G \in \mathfrak{R}^{n\times n}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} G + G^T - P & GA & GB \\ A^TG^T & P & 0 \\ B^TG^T & 0 & \mathbb{I} \end{bmatrix} > 0 \begin{bmatrix} W & C & D \\ C^T & P & 0 \\ D^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad (8)$$

iv) *There exist* $P = P^T \in \mathfrak{R}^{n\times n}$, $W = W^T \in \mathfrak{R}^{q\times q}$ *and* $G \in \mathfrak{R}^{n\times n}$, *such that:* $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} G + G^T - P & GA^T & GC^T \\ AG^T & P & 0 \\ CG^T & 0 & \mathbb{I} \end{bmatrix} 0, \begin{bmatrix} W & B^T & D^T \\ B & P & 0 \\ D & 0 & \mathbb{I} \end{bmatrix} 0, \quad (9)$$

Proof. See (Pipeleersa et al. 2009; Hilhorst et al. 2014).

Lemma 1.4 (Relaxed $\mathcal{H}_\infty$ performance). *Consider the system* (6). *For* $P = P^T > 0 \in \mathfrak{R}^{n\times n}$ *and* $G \in \mathfrak{R}^{n\times n}$, *the following statements are equivalent:*

(i) *A is stable and* $\| C(z\mathbb{I} - A)^{-1}BD \|_\infty < \gamma$

(ii) There exist P, *such that*

$$\begin{bmatrix} P & PA & PB & 0 \\ A^TP & P & 0 & C^T \\ B^TP & 0 & \gamma\mathbb{I} & D^T \\ 0 & C & D & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (10)$$

(iii) *There exist* P *and* G *such that*

$$\begin{bmatrix} G+G^T-P & GA & GB & 0 \\ A^TG^T & P & 0 & C^T \\ B^TG^T & 0 & \gamma\mathbb{I} & D^T \\ 0 & C & D & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (11)$$

(iv) *There exist P and G such that*

$$\begin{bmatrix} G+G^T-P & GA^T & GC^T & 0 \\ AG^T & P & 0 & B \\ CG^T & 0 & \gamma\mathbb{I} & D \\ 0 & B^T & D^T & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (12)$$

Proof. See (Pipeleersa et al., 2009; Hilhorst et al., 2014).

2 Descriptor LPV System

Let be a descriptor LPV system

$$\begin{aligned} E(\rho)x_{k+1} &= A(\rho)x_k + B(\rho)u_k \\ y_k &= C(\rho)x_k \end{aligned} \quad (13)$$

where $x \in \mathbb{R}^n$ is the vector of descriptor variable (instead of state vector), $\mathbb{E} \in \mathbb{R}^{m\times n}$, with $\text{rank}(\mathbb{E}) = r$ and $0 < r \le n$ for all ρ, which is called the descriptor matrix. $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$. $\rho_k \in \mathbb{R}^l$ is the varying parameter vector

$$\rho_k \triangleq \left[\rho_{1_k},\ \rho_{2_k},\ \cdots, \rho_{l_k}\right]^T \quad (14)$$

The descriptor matrix $E(\rho)$ is assumed to be rank-invariant, that is, $\text{rank}(\mathbb{E}) = r$ and $0 < r \le n$ for all admissible uncertainties ρ. This last asseveration is important for the robust admissibilization, when E is an uncertain matrix, (Mao 2012).

The range of each varying parameter ρ_i, $i = 1,2,\dots,l$, is given by

$$\rho_{i_k} \in \left[\underline{\rho}_i, \overline{\rho}_i\right], \quad \underline{\rho}_i < \overline{\rho}_i, \quad i = 1,2,\dots,l \quad (15)$$

Equation (15) represents a convex hull whose number of the vertices is $N = 2^l$. It corresponds to the operating region of the LPV system (13) and is also called the parameter box. The vertex set of the parameter box (15) is defined by

$$\Omega \triangleq \left\{\rho_k \in \mathbb{R}^l: \quad \rho_{i_k} = \underline{\rho}_i \text{ or } \overline{\rho}_i, \qquad i = 1,2,\dots,l\right\} \quad (16)$$

The descriptor LPV system (13) can be transformed into a polytopic model which is constructed by linearly combining LTI models in the vertex set Ω (16):

$$\begin{aligned} \sum_{j=1}^{N} \alpha_j(\rho) E_j x_{k+1} &= \sum_{j=1}^{N} \alpha_j(\rho) A_j x_k + \sum_{j=1}^{N} \alpha_j(\rho) B_j u_k \\ y_k &= \sum_{j=1}^{N} \alpha_j(\rho) C_j x_k \end{aligned} \quad (17)$$

Thus,

$$\left[E_j\ \ A_j\ \ B_j\ \ C_j\right] \triangleq \left[E(\rho^j)\ \ A(\rho^j)\ \ B(\rho^j)\ \ C(\rho^j)\right] \quad (18)$$

$$\rho^j \in \Omega,\ \ j = 1,2,\dots,N. \text{With } \sum_{j=1}^{N} \alpha_j = 1, \alpha_j \ge 0$$

In (Fujimori 2004) a general framework for to transform the Descriptor System (DS) (13) in form of descriptor polytopic models (17) is given.

Definition 2.1. *For (13) with $m = n$, if $\forall\rho$, $p(\mathrm{z}) = \det(\mathrm{z}E(\rho) - A(\rho))$ satisfies that $p(\mathrm{z}) \ne 0$, it is said that the pair $(E(\rho), A(\rho))$ is regular. Otherwise, it is called singular.*

Definition 2.2. *Consider the system (13) ($m = n$), with $\kappa = \deg(\det(\mathrm{z}E(\rho) - A(\rho))$. If $\kappa = r$ is said that the DS is of free impulse or causal.*

For the non-trivial case $E(\rho) \ne 0$, the property of free impulse it implies regularity. Considering that SD (13) free is not degenerate then it is an equivalent restrictive system given by

$$x_{1_{k+1}} = A_1 x_{1_k}, \qquad N x_{2_{k+1}} = x_{2_k}; \quad (19)$$

which it is causal if $N = 0$. Thus, the DS (13) has κ finite dynamic modes, $r - \kappa$ impulsive modes, and $n - r$ non-dynamic modes.

Theorem 2.1 (Stability of open-loop). *Let the system, with (13) the pair $(E(\rho), A(\rho))$ regular $\forall\rho$; and let $u = 0$.*

1) *For all ρ, the trivial solution $z_k = 0$ of the system is stable if and only if all the finite eigenvalues of $\lambda E(\rho) - A(\rho)$ are within the unit circle of the complex plane, and the eigenvalues at the boundary are simple.*
2) *For all ρ, the trivial solution $z_k = 0$ of the system is asymptotically stable if and only if all the finite eigenvalues of $\lambda E(\rho) - A(\rho)$ are within the unit circle of the complex plane. This means that the finite dynamic modes are asymptotically stable.*

Proof
See (Sjöberg, 2005; Duan, 2010).

Definition 2.3. *Consider the system (13). It is said that the DS is admissible if it is regular, free impulse (causal) and stable.*

In which it follows, the system *(13)* is considered regular and free impulse, consequently, the system admissibility corresponds to its stability.

Let consider the open-loop stability of system *(13)*.

Lemma2.2. *Let the system (13). That system is admissible if and only if there exists a matrix* $P = P^T > 0$ *such that the following inequalities are satisfied.*

$$E_j^T P\, E_j \geq 0, \;\; A_j^T P A_j - E_j^T P\, E_j < 0 \\ j = 1,2, \dots, N \tag{16}$$

Proof
Consider that exists $P = P^T > 0$ defining the Lyapunov function $V(x,\rho) = x^T E^T(\rho) P E(\rho) x$ such that $\forall\, \rho$, $E^T(\rho)PE(\rho) \geq 0$. Then, the stability condition establishes that $V_{k+1}(x,\rho) - V_k(x,\rho) < 0, \forall \rho$. Thus,

$$\begin{aligned} x_{k+1}^T E^T(\rho) P E(\rho) x_{k+1} - x_k^T E^T(\rho) P E(\rho) x_k &< 0 \\ x_k^T A^T(\rho) P A(\rho) x_k - x_k^T E^T(\rho) P E(\rho) x_k &< 0 \\ A^T(\rho) P A(\rho) - E^T(\rho) P E(\rho) &< 0 \\ \sum_{j=1}^{N} \alpha_j\, A_j^T P \sum_{j=1}^{N} \alpha_j\, A_j - \sum_{j=1}^{N} \alpha_j\, E_j^T P \sum_{j=1}^{N} \alpha_j\, E_j &< 0 \end{aligned}$$

Since $\sum_{j=1}^{N} \alpha_j = 1$, then
$A_j^T P A_j - E_j^T P E_j \;\; < 0, \qquad j = 1,2, \dots, N$
To condition that $E_j^T P E_j \geq 0$.

Condition (16), from Schur complement, can be written as

$$\begin{bmatrix} -E_j^T P E_j & A_j^T P \\ P A_j & -P \end{bmatrix} < 0, \qquad j = 1,2, \dots, N \tag{21}$$

Applying Lemma 2, see (Grman et al. 2005), then

$$\begin{bmatrix} -E_j^T P E_j & A_j^T G \\ G^T A_j & -G - G^T + P \end{bmatrix} < 0, \quad j = 1,2, \dots, N \tag{22}$$

The admissibility condition can also be established from dual system, given that the admissibility of the pair (E, A) is equivalent to the admissibility of (E^T, A^T), from the fact that $det(\mathrm{z}E - A) = det(\mathrm{z}E^T - A^T)$ and $deg\big(det(\mathrm{z}E - A)\big) = deg\big(det(\mathrm{z}E^T - A^T)\big)$, (see (Chadli, Darouach, 2012)), then

Lemma 2.3. *Let the system* (13). *That system is admissible if and only if there exists a matrix* $P = P^T > 0$ *such that the following equivalent linear inequalities are satisfied.*

i)
$$E_j P\, E_j^T \geq 0, \; A_j P A_j^T - E_j P\, E_j^T < 0, j = 1,2, \dots, N \tag{23}$$

ii)
$$\begin{bmatrix} -E_j P E_j^T & A_j G \\ G^T A_j^T & -G - G^T + P \end{bmatrix} 0, \qquad j = 1,2, \dots, N \tag{24}$$

As it can be noticed in (23), the inequality $E_j^T P\, E_j \geq 0$ is not strict which results in difficulty in computation. In order to remove such inequality and establish new strict matrix inequality conditions, in (Zhang et al., 2008) a new condition is proposed:

Theorem 2.4 (Stability of open-loop: (G. Zhang, Xia, and Shi 2008)). *The system (13) is admissible if and only if there exists a positive definite matrix* $Q \in \mathbb{R}^{n\times n}$ *and a symmetric matrix* $S \in \mathbb{R}^{(n-r)\times(n-r)}$ *such that*

$$A_j^T \left(Q - \overline{E}_j^T S \overline{E}_j \right) A_j - E_j^T Q E_j < 0, \;\; j = 1,2, \dots, N \tag{25}$$

with $\overline{E}_j$ *denotes a matrix with the properties of* $N(\overline{E}_j) = R(E_j)$, *i.e.* $\overline{E}_j E_j = 0$, *and* $\overline{E}_j\, \overline{E}_j^T > 0$.

Proof
See (Zhang, Xia, Shi, 2008).

The matrix $\overline{E} \in \mathbb{R}^{(n-r)\times n}$, which is of full column ranks, it is composed of base of ker E.

Using the Schur complement, then:

Lemma 2.5. *The system* (13) *is admissible if and only if there exists a positive definite matrix* $Q \in \mathbb{R}^{n\times n}$, *a symmetric matrix* $S \in \mathbb{R}^{(n-r)\times(n-r)}$ *and a matrix* $G \in \mathbb{R}^{n\times n}$ *such that for* $j = 1,2, \dots, N$

$$\begin{bmatrix} -E_j^T Q E_j & A_j^T G \\ G^T A_j & -G - G^T + Q - \overline{E}_j^T S \overline{E}_j \end{bmatrix} < 0, \tag{26}$$

Proof
As it is known and since $Q - \overline{E}_j^T S \overline{E}_j$ is positive definite, (25) can be can be described by (for all $j = 1,2, \dots, N$)

$$\begin{bmatrix} -E_j^T Q E_j & A_j^T\left(Q-\overline{E}_j^T S\overline{E}_j\right) \\ \left(Q-\overline{E}_j^T S\overline{E}_j\right)^T A_j & -\left(Q-\overline{E}_j^T S\overline{E}_j\right) \end{bmatrix} < 0, \quad (27)$$

In order to recover Theorem 2.4 (*necessity*), for $G = G^T = Q-\overline{E}_j^T S\overline{E}_j$, the inequality (27) holds. For the *sufficiency*, the inequality (26) is assumed feasible. Hence $G+G^T > Q-\overline{E}_j^T S\overline{E}_j > 0$ for all $j = 1,3,\dots,N$. If $Q-\overline{E}_j^T S\overline{E}_j$ is positive definite, $\forall\, j = 1,3,\dots,N$, then the inequality $\left(Q-\overline{E}_j^T S\overline{E}_j - G\right)^T\left(Q-\overline{E}_j^T S\overline{E}_j\right)^{-1}\left(Q-\overline{E}_j^T S\overline{E}_j - G\right) \geq 0$ is satisfied. Therefore, the following condition can be established: $G^T\left(Q-\overline{E}_j^T S\overline{E}_j\right)^{-1} G \geq G+G^T-\left(Q-\overline{E}_j^T S\overline{E}_j\right)$, which yields ($\forall\, j = 1,2,\dots,N$)

$$\begin{bmatrix} -E_j^T Q E_j & A_j^T G \\ G^T A_j & -G^T\left(Q-\overline{E}_j^T S\overline{E}_j\right)^{-1} G \end{bmatrix} < 0,$$

Since G is nonsingular, matrix inequality is multiplied by the right by the matrix $T := diag[\mathbb{I}, G^{-1}\left(Q-\overline{E}_j^T S\overline{E}_j\right)]$ and by the left by T^T from which the matrix inequality (27) is recovered.

The condition for the equivalent dual system is given by ($\forall\, j = 1,2,\dots,N$):

$$\begin{bmatrix} -E_j Q E_j^T & A_j G \\ G^T A_j^T & -G-G^T+Q-\overline{\overline{E}}_j S\overline{E}_j^T \end{bmatrix} < 0, \quad (28)$$

to condition that $\overline{E}_j^T E_j^T = 0$ and $\overline{E}_j^T \overline{\overline{E}}_j > 0$ for all $j = 1,2,\dots,N$.

A new extended and improved condition has been presented in (Mohammed Chadli and Darouach 2012), which is described in the following theorem:

Theorem 2.6 (Stability of open-loop: (Mohammed Chadli and Darouach 2012)). *The system* (13) *is admissible if and only if the following equivalent statements hold.*

(i) *There exists a matrix* $P = P^T$ *satisfying the LMI* (28).

(ii) *There exist matrices* $Q > 0$ *and* $S = S^T$ *satisfying the LMI* (25).

(iii) *There exist matrices* $Q > 0$ *and* $S = S^T$ *satisfying the LMI*

$$A_j\left(Q-\overline{\overline{E}}_j S\overline{E}_j^T\right)A_j^T - E_j Q E_j^T < 0, j = 1,2,\dots,N \quad (29)$$

with $\overline{E}_j^T E_j^T = 0$ *and* $\overline{E}_j^T \overline{\overline{E}}_j > 0$.

(iv) *There exist matrices* $Q > 0$, $S = S^T$, F *and* G *satisfying,* $\forall\, j = 1,2,\dots,N$, *the following LMI*

$$\begin{bmatrix} -E_j^T Q E_j + A_j^T F^T + F A_j & -F + A_j^T G^T \\ -F^T + G A_j & -G-G^T+Q-\overline{E}_j^T S\overline{E}_j \end{bmatrix} < 0 \quad (30)$$

(v) *There exist matrices* $Q > 0$, $S = S^T$, F *and* G *satisfying,* $\forall\, j = 1,2,\dots,N$, *the following LMI*

$$\begin{bmatrix} -E_j Q E_j^T + A_j F^T + F A_j^T & -F + A_j G^T \\ -F^T + G A_j^T & -G-G^T+Q-\overline{\overline{E}}_j S\overline{E}_j^T \end{bmatrix} < 0 \quad (31)$$

Proof

See (Chadli, Darouach, 2012).

2.1 Numerical Examples

Consider the matrices, (Mohammed Chadli and Darouach 2012):

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2.5 & 1 \\ 1.7 & 0.8 \end{bmatrix}; \qquad \overline{E} = [0 \quad 1]$$

Applying Theorem 2.4, the following matrices are obtained:

$$Q = \begin{bmatrix} 0.1857 & -0.1462 \\ -0.1462 & 1.2582 \end{bmatrix}, \qquad S = 1.4142$$

For next example, the following matrices are considered, (G. Zhang, Xia, and Shi 2008):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2000 & 0.1000 & 0.1000 \\ 0.1000 & 0 & 0 \\ 0 & 0 & 0.1000 \end{bmatrix};$$

$$\overline{E} = [0 \quad 0 \quad 50]$$

From Theorem 2.4:

$$Q = 1.0e+03 \times \begin{bmatrix} 1.0957 & 0.0106 & -0.0005 \\ 0.0106 & 1.0749 & -0.0002 \\ -0.0005 & -0.0002 & 1.0696 \end{bmatrix},$$
$$S = 43.65$$

The conditions for admissibility that are characterized in terms of strict LMIs allow to design stabilizing controllers for descriptor LPV systems.

2.2 Robust Stabilization by State Feedback Control

In order to design appropriate controllers, it is necessary to evaluate the characterization of the controllability condition for DS.

Definition 2.4. *A linear DS is said controllable if, for any time $k > 0$, $x_0 \in \mathbb{R}^n$ and $x_f \in \mathbb{R}^n$, there exits a control u such that $x_k = x_f$*

Lemma 2.7 (Controllability of Descriptor LPV System). *Consider system* (13)*, which is said controllable if and only if, for all ρ:*

$$rank\ [zE(\rho) - A(\rho)\ \ B(\rho)] = n, \forall z \in \mathbb{C},\ z\ \ finite\ \ and$$
$$rank\ [E(\rho)\ \ B(\rho)] = n \tag{32}$$

Proof

See (Sjöberg, 2005; Scholz, 2015).

In (32), the first condition talks about the controllability of the finite dynamic modes. The second condition is related to the controllability of the impulsive modes.

The Definition 2.3 and the Lemma 2.7 allow to establish conditions for the control of DS in the sense of its stabilization (Chaabane et al. 2011; Mohammed Chadli and Darouach 2012). Thus, it is necessary to consider the controllability condition referred to the stability or stabilization of the DS type LPV.

Indeed, let be system (13), for all ρ:

1) The triplet $\big(E(\rho), A(\rho), B(\rho)\big)$ is said that the system has stabilizable finite dynamics and impulse controllable if a matrix $\mathbb{K}$ exists such that the pair $(E(\rho), A(\rho) + B(\rho)\mathbb{K})$ is admissible.
2) The triplet $\big(E(\rho), A(\rho), C(\rho)\big)$ is said that the system is of finite dynamics detectable and impulse observable if a matrix $\mathbb{L}$ exists such that the pair $\big(E(\rho), A(\rho) + \mathbb{L}C(\rho)\big)$ is admissible.

On the other hand, when the index of the system; or the maximum size of the Jordan blocks in the canonical form *Weierstraß* of the matrix pair (E, A); is not greater than 1 and the pair (E, A) is regular, the algebraic part (or the associated redundant variables) can be eliminated in (13), resulting in a standard linear system of reduced order, (with a non-square matrix $\mathbb{E}$, which has a generalized inverse). Conversely, systems with an index higher than 1 may lose the causality for some insufficiently smooth inputs.

Let consider that system (13) is controllable for all ρ, thus, the robust admissibilization of system (13) is considered by means of a state feedback control. Then, consider the control u_k given by

$$u_k = \boldsymbol{K} x_k, \tag{33}$$

then, the system in cloed-loop is given by

$$E(\rho)x_{k+1} = (A(\rho) + B(\rho)\boldsymbol{K})x_k \tag{34}$$

From the Theorem 2.6 on the equivalent dual system, the following result is determined:

Lemma 2.8. *The system (13) is robustly admissible via a state feedback controller (33), if and only if there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, a symmetric matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ and a matrix $G \in \mathbb{R}^{n\times n}$, a matrix $F \in \mathbb{R}^{n\times n}$ and the matrices $X \in \mathbb{R}^{q\times n}$, $Y \in \mathbb{R}^{q\times n}$, such that, $\forall\ j = 1,2, \dots, N$:*

$$\begin{bmatrix} -E_j Q E_j^T + A_j F^T + B_j X + F A_j^T + X^T B_j^T & -F + A_j G^T + B_j Y \\ -F^T + G A_j^T + Y^T B_j^T & -G - G^T + Q - \overline{\overline{E}}_j S \overline{E}_j^T \end{bmatrix} < 0, \tag{35}$$

where the feedback gain is given by

$$\boldsymbol{K} = Y(G^T)^{-1} \tag{36}$$

Proof

Applying Lemma 2.5 with respect to the condition defined by (28), using the dynamic matrix of the closed-loop $A_j + B_j\boldsymbol{K}$, then the change of variables $Y = \boldsymbol{K}G^T$ and $X = \boldsymbol{K}F^T$ must be used.

2.2.1 Numerical Example

1) For this example, the following matrices are considered:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0.1 & 0.1 \\ 0.1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix}; \quad \overline{E}^T = \begin{bmatrix} 0 & 0 & 20 \end{bmatrix}$$

Applying the Lemma 2.8, the resulting matrices necessary to calculate the feedback gain are:

$$G = \begin{bmatrix} 53.13 & -3.523 & -120.00 \\ 3.31 & 53.69 & 0.69 \\ 119.64 & -0.71 & -546.45 \end{bmatrix},$$

$$Y = 1.0e + 03 \times \begin{bmatrix} -1.73 & -0.18 & -5.37 \\ 0.84 & -0.09 & 2.60 \end{bmatrix}$$

Thus,

$$\boldsymbol{K} = \begin{bmatrix} -20.6497 & -2.1042 & 5.3028 \\ 9.8561 & -2.1983 & -2.5945 \end{bmatrix}$$

The dynamic matrix of closed-loop is

$$\boldsymbol{A} = \begin{bmatrix} 0.9206 & -0.3302 & 0.3708 \\ 0.0062 & -0.6501 & 0.0114 \\ 1.9712 & -0.4397 & 0.4811 \end{bmatrix}$$

Thus, the finite eigenvalues of $\lambda E - \boldsymbol{A}$ are: $\lambda_1 = -0.6109$, $\lambda_2 = -0.6276$

2) For next example, let consider the descriptor LPV system:

$$\begin{bmatrix} 2+\rho_1 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} = \begin{bmatrix} 1+2\rho_2 & 1 \\ 2 & -\rho_1 \end{bmatrix} x_k + \begin{bmatrix} -\rho_1 \\ 1 \end{bmatrix} u_k$$

with $\rho_1 \in [-1 \;\; 1]$ and $\rho_2 \in [-1 \;\; 1]$.
In order to apply the Lemma 2.8, the following matrices are considered:

$$\overline{E}_1^T = \overline{E}_3^T = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad \overline{E}_2^T = \overline{E}_4^T \begin{bmatrix} \frac{1}{3} & -1 \end{bmatrix}$$

which are in correspondence to vertex matrices E_j for $j = 1, \dots, 4$. Thus, for $F = 0$ the results obtained are

$$S = 178.3975, \quad G = \begin{bmatrix} 6.3429 & 27.8754 \\ 2.2215 & -1.6985 \end{bmatrix},$$

$$Y = [19.3438 \quad -44.4202]$$

Therefore, the feedback gain is
$\boldsymbol{K} = [-16.5803 \quad 4.4667]$

Therefore, for the pair $(E_j, \boldsymbol{A}_j)$, $J = 1,2,\dots,4$, where $\boldsymbol{A}_j$ is the dynamic matrix of closed-loop, the finite poles are: $-0.8181, 0.1388, 0.2727, 0.6939$, respectively. In order to evaluate the robust admissibility for the closed-loop system, Figure 1 and 2 shows the distribution of the poles and its projection in the complex plane as results of the parametric variations. Easily it is possible to be concluded that the system in closed-loop is admissible robustly.

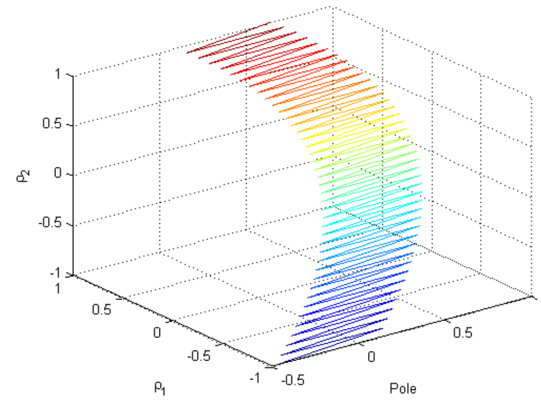

Fig. 1. The pole distribution.
Distribución de polos**.**

3) Let be system with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} -.25 & 0.1\rho & 0.01\rho \\ -0.5 & 0.5 - 0.01\rho & 2 - 0.01\rho \\ 0.75 & -1 + 0.005\rho & -1.5 + 0.005\rho \end{bmatrix},$$

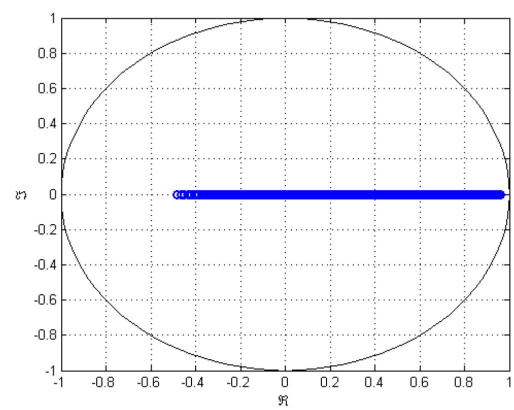

Fig. 2. The pole projection in the complex plane.
Proyección de polos en el plano complejo.

with $\rho_1 \in [-1 \quad 1]$. Considering $\overline{E}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, then

$$(G^T)^{-1} = 1.0e + 11 \times \begin{bmatrix} -0.0815 & 0.0828 & 0.3264 \\ -0.0000 & 0.0000 & 0.0000 \\ 0.2872 & -0.2919 & -1.1499 \end{bmatrix},$$

$$Yf = [-0.0000 \quad 0.1524 \quad -0.0000],$$
$$S = 6.4399e + 08$$

Therefore, the feedback gain corresponding is
$\boldsymbol{K} = [-0.6002 \quad 0.2868 \quad -1.3517]$

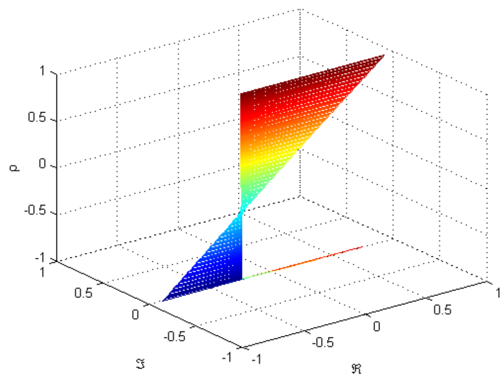

Fig. 3. The closed-loop poles.
Polos del lazo cerrado.

For this particular case, the obtained finite stable poles are real.

4) Let consider the following descriptor LPV system with:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A(= \begin{pmatrix} 0 & 1 & 0 & \rho_1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0.75 & 0.25 & \rho_2 \end{pmatrix}, \quad (37)$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0.895 & 1 \\ -1 & 1 \end{pmatrix}$$

where: $\rho_1 \in [0 \quad 0.5]$; $\rho_2 \in [0.5 \quad 1.0]$.

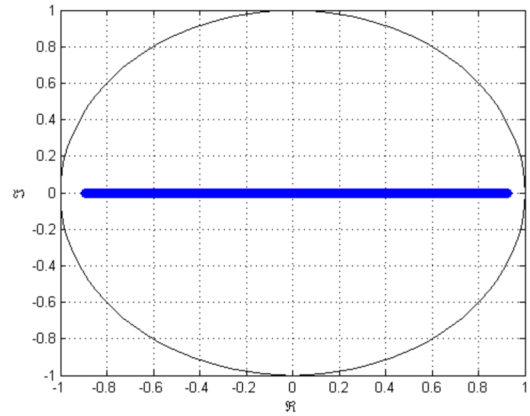

Fig. 4. The projection in the complex plane.
Proyección en el plano complejo.

For $\overline{E}^T = [0 \quad 0 \quad 0 \quad 1]$, then the following matrices are obtained:

$$G = 1.0e + 07 \times \begin{bmatrix} 1.6794 & -0.4578 & 0.3591 & 0.0000 \\ -0.6469 & 0.7345 & -0.3318 & -0.0000 \\ 0.2381 & -0.2044 & 0.6916 & 0.0000 \\ -0.0017 & 0.0026 & -0.0047 & 0.0000 \end{bmatrix},$$

$$Y = 1.0e + 06 \times \begin{bmatrix} 8.27 & -0.72 & -0.42 & 0.003 \\ -5.99 & 1.07 & -2.99 & 0.01 \end{bmatrix},$$

$$S = 8.1032e + 03$$

Consequently, the feedback gain is

$$\boldsymbol{K} = 1.0e + 11 \times \begin{bmatrix} 0.0000 & 0.0000 & -0.0000 & -9.8756 \\ -0.0000 & -0.0000 & -0.0000 & -9.8756 \end{bmatrix}$$

For the closed-loop system, Figure 5 and 6 shows the magnitude of the poles with respect to the parametric variations, and also the location of the poles in the complex plane.

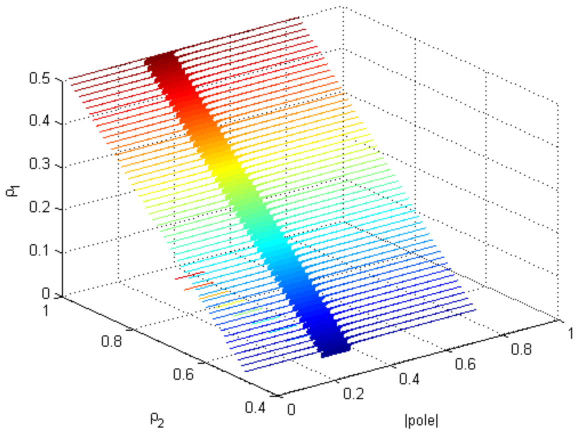

Fig. 5. The pole distribution for the system (37).
Distribución de polos para el sistema (37).

2.3 Robust admissibilization with pole location

In this section, the idea is to present an extension of the results in (Peaucelle et al. 2000; Krokavec and Filasová 2019) to the case of descriptor LPV systems. Thus, the more generalized *D*-stable region is considered, defining the ellipse parameters in order to obtain the *D*-stability area.

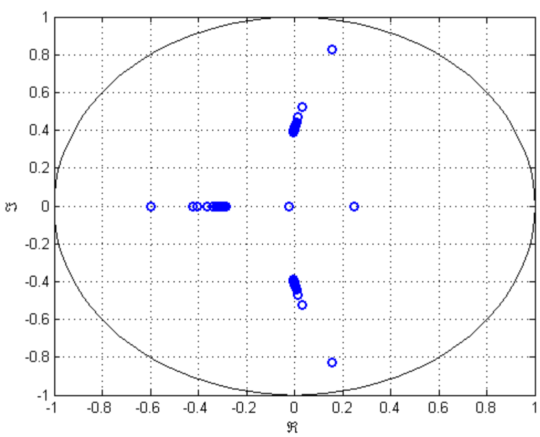

Fig. 6. The pole projection for the system (37).
Proyección de polos para el sistema (37).

In (Krokavec, Filasová, 2019) the Lypunov-Krasovskii theorem is used, then the closed-loop system pole clustering is tailored via a finite set of LMIs with expansion of the Lyapunov matrix inequality. Thus, let consider the stability of autonomous descriptor system given by

$$Ex_{k+1} = \frac{A - \sigma\mathbb{I}}{r} x_k$$

where $\sigma,\ r \in \mathbb{R}$ and $y \leq b < r, \sigma + r < 1$, being defined the analytic function of an ellipsoid in the $\boldsymbol{Z}-$ plane, (Krokavec and Filasová 2019):

$$\frac{(x-\sigma)^2}{r^2} + \frac{y^2}{b^2} < 1$$

Let be the Lyapunov function $V(x_k) = x_k^T E^T P E x_k > 0$, then from the Lyapunov-Krasovskii theorem

$$x_{k+1}^T E^T P E x_{k+1} - x_k^T E^T P E x_k \leq$$

$$-\frac{y^2}{r^2}\left(\frac{r^2}{b^2} - 1\right) x_k^T E^T P E x_k < 0,$$

then

$$(A - \sigma\mathbb{I})^T P (A - \sigma\mathbb{I}) - r^2 E^T P E + y^2\left(\frac{r^2}{b^2} - 1\right) E^T P E < 0 \tag{39}$$

$$\left[-r^2 + y^2\left(\frac{r^2}{b^2} - 1\right)\right] E^T P E - \sigma P A - \sigma A^T P + A^T P A + \sigma^2 P < 0 \tag{40}$$

Lemma 2.9. *Consider system (38), then the matrix the pair (E, A) is* $\boldsymbol{D}$ *-stable if and only if for given positive scalars* $\sigma,\ r \in \mathbb{R}$ *and* $y \leq b < r,\ \sigma + r < 1$ *there exist* $Q \in \mathbb{R}^{n\times n},\ Q = Q^T > 0$, *such that*

i)

$$\begin{bmatrix} \left(-r^2 + y^2\left(\frac{r^2}{b^2} - 1\right)\right) E^T Q E & A^T Q - \sigma Q \\ QA - \sigma Q & -Q \end{bmatrix} < 0 \tag{41}$$

ii)

$$\begin{bmatrix} \left(-r^2 + y^2\left(\frac{r^2}{b^2} - 1\right)\right) E Q E^T & AQ - \sigma Q \\ QA^T - \sigma Q & -Q \end{bmatrix} < 0 \tag{42}$$

Proof

The Schur complement is applied for the inequality matrix (39). The item *ii*) is the dual of *i*).

- It is possible be noted that for limit case ($y = b$), the the D -stability region is strictly given by the area inside the ellipse.
- If $r = b$, the stable poles are located in a particular region, D -circle stability area, which is shown in the Figure 7. There, r is the radius and the center is defined by $(\sigma, 0)$, with $|\sigma| + r \leq 1$ and $|\sigma| < 1$.
- For the previous case, if $r = 1$ and $\sigma = 0$, it is evident that D-stability region is the open unit disc of the complex $\boldsymbol{Z}$ -plane.

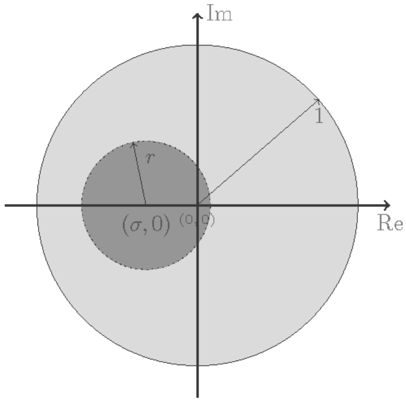

Fig. 7. Location of stable poles in a D -circle region.
Ubicación de polos estables en una región D-circulo.

This is derived by considering that the dynamic matrix corresponds to the matrix $\frac{A-\sigma\mathbb{I}_n}{r}$. This condition is extended for descriptor LPV system as follows:

Lemma 2.10. *The system (13) is admissible with finite poles in a* D*-stability region defined by the parameters $\sigma,\ r\in\mathbb{R}$ and $y\le b<r$, $\sigma+r<1$, if and only if there exists a positive definite matrix $Q\in\mathbb{R}^{n\times n}$, a symmetric matrix $S\in\mathbb{R}^{(n-r)\times(n-r)}$ and a matrix $G\in\mathbb{R}^{n\times n}$, with $G+G^T>0$, satisfying, $\forall j=1,2,\ldots N$, the following LMI of the equivalent statements:*

1)
$$\begin{bmatrix}\left(-r^2+y^2\left(\frac{r^2}{b^2}-1\right)\right)E_j^TQE_j & A_j^TG-\sigma G\\ G^TA_j-\sigma G^T & -G-G^T+Q-\overline{E}_j^TS\overline{E}_j\end{bmatrix}<0, \tag{43}$$

2)
$$\begin{bmatrix}\left(-r^2+y^2\left(\frac{r^2}{b^2}-1\right)\right)E_jQE_j^T & A_jG-\sigma G\\ G^TA_j^T-\sigma G^T & -G-G^T+Q-\overline{\overline{E}}_j^TS\overline{E}_j^T\end{bmatrix}<0, \tag{44}$$

__**Proof**__

Simply, from Lemma 2.9 and (26), A_j is replaced by $\frac{A_j-\sigma\mathbb{I}_n}{r}$ then, by means of algebraic manipulation, the matrix inequality (43) and (44) are obtained.

Remark 2.1. *From (40), the LMIs in the Lemma 2.9 are equivalents to*

i)
$$\begin{bmatrix}\Gamma & A^TQ+\sigma Q\\ QA+\sigma Q & -Q\end{bmatrix}<0 \tag{45}$$

with $\Gamma=\left(-r^2+y^2\left(\frac{r^2}{b^2}-1\right)\right)E^TQE-\sigma QA-\sigma A^TQ.$

ii)
$$\begin{bmatrix}\Upsilon & AQ+\sigma Q\\ QA^T+\sigma Q & -Q\end{bmatrix}<0 \tag{46}$$

with $\Upsilon=\left(-r^2+y^2\left(\frac{r^2}{b^2}-1\right)\right)EQE^T-\sigma QA^T-\sigma AQ.$

3 Bounded Real Lemma for Descriptor LVP System

The Bounded Real Lemma (BRL) allows to establish a characterization of the $\mathcal{H}_\infty$-norm for dynamic systems as a LMI. Immediately a strict LMI condition under which descriptor LPV system is admissible with an $\mathcal{H}_\infty$-norm smaller than a prescribed positive number. Thus, the following results are obtained from (G. Zhang, Xia, and Shi 2008) and (Mohammed Chadli and Darouach 2012). Let consider the descriptor LPV system type polytopic:

$$\begin{aligned}E(\rho)x_{k+1}&=A(\rho)x_k+B_\omega(\rho)\omega_k+B(\rho)u_k\\ y_k&=C(\rho)x_k+D(\rho)\omega_k\end{aligned} \tag{47}$$

where $\omega_k\in\mathbb{R}^d$ are perturbation signals. The matrices $B_\omega(\rho)$, $D(\rho)$ have appropriated dimensions. Then, the transfer function $H_{\omega y}(\rho,\mathrm{z})=C(\rho)\big(\mathrm{z}E(\rho)-A(\rho)\big)^{-1}B_\omega(\rho)+D(\rho)$ is defined.

Lemma 3.1. *The discrete-time descriptor LPV system (47) is admissible and satisfies $\|H_{\omega y}\|_\infty<\gamma$, if and only if there exists a positive definite $Q\in\mathbb{R}^{n\times n}$ and a symmetric matrix $S\in\mathbb{R}^{(n-r)\times(n-r)}$ satisfying, $\forall j=1,2,\ldots N$, the following LMI*

$$\begin{bmatrix}-E_j^TQE_j+A_j^T(Q-\overline{E}_j^TS\overline{E}_j)A_j+C_j^TC_j & C_j^TD_j+A_j^T(Q-\overline{E}_j^TS\overline{E}_j)B_{\omega j}\\ D_j^TC_j+B_{\omega j}^T(Q^T-\overline{E}_jS\overline{E}_j^T)A_j & -\gamma^2\mathbb{I}+D_j^TD_j+B_{\omega j}^T(Q-\overline{E}_j^TS\overline{E}_j)B_{\omega j}\end{bmatrix}<0 \tag{48}$$

__**Proof**__

See (Zhang et al., 2008; Chadli, Darouach, 2012).

Theorem 3.2. *The discrete-time descriptor LPV system (47) is admissible and satisfies $\|H_{\omega y}\|_\infty<\gamma$, if and only if there exists a positive definite $Q\in\mathbb{R}^{n\times n}$, a symmetric* *matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ and the matrices $\mathfrak{F}$, $\mathfrak{G}$ satisfying, $\forall j = 1,2,\dots N$, the following LMI of the equivalent statements:*

i) (48) is holds.

ii)

$$\begin{bmatrix} -\mathfrak{E}_j^T \mathbb{Q}\mathfrak{E}_j + \mathfrak{F}\mathfrak{A}_j + \mathfrak{A}_j^T\mathfrak{F}^T & -\mathfrak{F} + \mathfrak{A}_j^T\mathfrak{G}^T \\ -\mathfrak{F}^T + \mathfrak{G}\mathfrak{A}_j & \mathbb{Q} - \mathfrak{G} - \mathfrak{G}^T - \overline{\mathfrak{E}}_j^T \mathbb{S}\overline{\mathfrak{E}}_j \end{bmatrix} < 0 \quad (49)$$

iii)

$$\begin{bmatrix} -\mathfrak{E}_j \mathbb{Q}\mathfrak{E}_j^T + \mathfrak{F}\mathfrak{A}_j^T + \mathfrak{A}_j\mathfrak{F}^T & -\mathfrak{F} + \mathfrak{A}_j\mathfrak{G}^T \\ -\mathfrak{F}^T + \mathfrak{G}\mathfrak{A}_j^T & \mathbb{Q} - \mathfrak{G} - \mathfrak{G}^T - \overline{\overline{\mathfrak{E}}}_j \mathbb{S}\overline{\mathfrak{E}}_j^T \end{bmatrix} < 0 \quad (50)$$

with

$$\mathfrak{A}_j = \begin{bmatrix} A_j & B_{\omega j} \\ C_j & D_j \end{bmatrix}, \quad \mathbb{Q} = \begin{bmatrix} Q & 0 \\ 0 & \mathbb{I}_p \end{bmatrix},$$

$$\mathfrak{E}_j = \begin{bmatrix} E_j & 0 \\ 0 & \mathbb{I}_{p\times d} \end{bmatrix}, \quad \mathbb{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

Proof

See (G. Zhang, Xia, and Shi 2008; Mohammed Chadli and Darouach 2012).

4 Main results: $\mathcal{H}_2 - \mathcal{H}_\infty$ -Control for Descriptor LVP System

First, let us look again at the strict LMI characterization of the $\mathcal{H}_2$-norm for descriptor LVP systems. A first strict LMI condition for $\mathcal{H}_2$ control of LTI descriptor systems is presented in (Ikeda, LEE, and Uezat 2000). This condition is based on Theorem 2.4.

Let consider (47). Then, $\| H_{\omega y} \|_2^2 = tr\,[D^T(\rho)D(\rho) + B_\omega^T(\rho)QB_\omega(\rho)]$, such that $Q = Q^T > 0$ and $A^T(\rho)QA(\rho) - E^T(\rho)QE(\rho) + C^T(\rho)C(\rho) < 0$.

Theorem 4.1. *The discrete-time descriptor LPV system (47) is admissible and satisfies $\| H_{\omega y} \|_2^2 < \mu$, if and only if there exists a positive definite $Q \in \mathbb{R}^{n\times n}$, a symmetric matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ and the matrices $W \in \mathbb{R}^{d\times d}$ with $tr(W) < \mu$, G satisfying, $\forall j = 1,2,\dots N$, the following LMI of the equivalent statements:*

i)

$$\begin{bmatrix} E_j^T QE_j & A_j^T Q & C_j^T \\ QA_j & Q & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0, \begin{bmatrix} W & B_{\omega j}^T Q & D_j^T \\ QB_{\omega j} & Q & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (51)$$

ii)

$$\begin{bmatrix} E_j QE_j^T & A_j Q & B_{\omega j} \\ QA_j^T & Q & 0 \\ B_{\omega j}^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad \begin{bmatrix} W & C_j Q & D_j \\ QC_j^T & Q & 0 \\ D_j^T & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (52)$$

iii)

$$\begin{bmatrix} E_j^T QE_j & A_j^T G & C_j^T \\ G^T A_j & G + G^T - Q + \overline{E}_j^T S\overline{E}_j & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0$$

$$\begin{bmatrix} W & B_{\omega j}^T G & D^T \\ G^T B_{\omega j} & G + G^T - Q + \overline{E}_j^T S\overline{E}_j & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (53)$$

iv)

$$\begin{bmatrix} E_j QE_j^T & A_j G & B_{\omega j} \\ G^T A_j^T & G + G^T - Q + \overline{\overline{E}}_j S\overline{E}_j^T & 0 \\ B_{\omega j}^T & 0 & \mathbb{I} \end{bmatrix} > 0$$

$$\begin{bmatrix} W & C_j G & D \\ G^T C_j^T & G + G^T - Q + \overline{\overline{E}}_j S\overline{E}_j^T & 0 \\ D_j^T & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (54)$$

Proof

Since *ii)* is the dual one of *i)*, then its equivalence is evident. Equivalence between *iii)* and *i)* is as it follows: consider the matrix $= R^T = (Q - \overline{E}_j^T S\overline{E}_j) > 0$, with $\overline{E}_j E_j = 0$ and $\overline{E}_j\,\overline{E}_j^T > 0$, such that $\| H_{\omega y} \|_2^2 = tr\,[D^T(\rho)D(\rho) + B^T(\rho)RB(\rho)]$, and $A^T(\rho)RA(\rho) - E^T(\rho)RE(\rho) + C^T(\rho)C(\rho) < 0$. Thus, *iii)* must be equivalent to

$$\begin{bmatrix} E_j^T QE_j & A_j^T R & C_j^T \\ RA_j & R & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0, \begin{bmatrix} W & B_{\omega j}^T R & D_j^T \\ RB_{\omega j} & R & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (55)$$

which is equivalent to

$$\begin{bmatrix} E_j^T QE_j & A_j^T(Q - \overline{E}_j^T S\overline{E}_j) & C_j^T \\ (Q - \overline{E}_j^T S\overline{E}_j)^T A_j & \left(Q - \overline{E}_j^T S\overline{E}_j\right) & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0,$$

$$\begin{bmatrix} W & B_{\omega j}^T(Q-\overline{E}_j^T S\overline{E}_j) & D_j^T \\ (Q-\overline{E}_j^T S\overline{E}_j)^T B_{\omega j} & \left(Q-\overline{E}_j^T S\overline{E}_j\right) & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0$$

From (53), for $G = G^T = R$, the inequality (55) holds. For the sufficiency, the inequality (53) is assumed feasible. Thus, let $G + G^T > R$, then $(G - R)^T R^{-1}(G - R) \geq 0$; therefore $G^T R^{-1} G \geq G + G^T - R$. Consequently,

$$\begin{bmatrix} E_j^T Q E_j & A_j^T G & C_j^T \\ G^T A_j & G^T R^{-1} G & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0$$

$$\begin{bmatrix} W & B_{\omega j}^T G & D_j^T \\ G^T B_{\omega j} & G^T R^{-1} G & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (56)$$

Let consider a matrix $\mathrm{T} = diag[\mathbb{I}, G^{-1}R, \mathbb{I}]$, then the inequalities in (56) are multiplied by the right by T and the left by T^T; so that (55) is obtained, where R can be replaced by $Q - \overline{E}_j^T S\overline{E}_j$. Finally, *iv)* is the dual equivalent of *iii)*.

Lemma 4.2. *The discrete-time descriptor LPV system (47) is admissible and satisfies* $\| H_{\omega y} \|_2^2 < \mu$, *if and only if:*

i) *there exists a positive definite* $Q \in \mathbb{R}^{n\times n}$, *a symmetric matrix* $S \in \mathbb{R}^{(n-r)\times(n-r)}$ *and the matrices* $W \in \mathbb{R}^{d\times d}$ *with* tr$(W) < \mu$, *and* G *satisfying,* $\forall j = 1,2,\ldots N$, *the following LMI*

$$\begin{bmatrix} E_j^T Q E_j & A_j^T Q & C_j^T \\ Q A_j & Q & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0, \begin{bmatrix} W & B_{\omega j}^T Q & D_j^T \\ Q B_{\omega j} & Q & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (57)$$

ii) *there exists a positive definite* $Q \in \mathbb{R}^{n\times n}$, *a symmetric matrix* $S \in \mathbb{R}^{(n-r)\times(n-r)}$ *and the matrices* $W \in \mathbb{R}^{d\times d}$ *with* tr$(W) < \mu$, G *and* F *satisfying,* $\forall j = 1,2,\ldots N$, *the following LMI*

$$\begin{bmatrix} E_j^T Q E_j - F A_j - A_j^T F^T & -F + A_j^T G & C_j^T \\ -F^T + G^T A_j & \Xi & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (58)$$

$$\begin{bmatrix} W & B_{\omega j}^T G & D^T \\ G^T B_{\omega j} & \Xi & 0 \\ D_j & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (59)$$

with $\Xi = G + G^T - Q + \overline{E}_j^T S\overline{E}_j$.

iii) *there exists a positive definite* $Q \in \mathbb{R}^{n\times n}$, *a symmetric matrix* $S \in \mathbb{R}^{(n-r)\times(n-r)}$ *and the matrices* $W \in \mathbb{R}^{p\times p}$ *with tr* $(W) < \mu$, G *and* F *satisfying,* $\forall j = 1,2,\ldots N$, *the following LMI*

$$\begin{bmatrix} E_j Q E_j^T - F A_j^T - A_j F^T & -F + A_j G & B_{\omega j} \\ -F^T + G^T A_j^T & \Phi & 0 \\ B_{\omega j}^T & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (60)$$

$$\begin{bmatrix} W & C_j G & D \\ G^T C_j^T & \Phi & 0 \\ D_j^T & 0 & \mathbb{I} \end{bmatrix} > 0 \qquad (61)$$

Where $\Phi = G + G^T - Q + \overline{\overline{E}}_j S\overline{E}_j^T$.

<u>Proof</u>

The proof is based on establishing equivalence between the LMI (53) and its corresponding in (51), (Zhang et al., 2008; Chadli, Darouach, 2012). Thus, consider the matrix $= R^T = (Q - \overline{E}_j^T S\overline{E}_j) > 0$, with $\overline{E}_j E_j = 0$ and $\overline{E}_j \, \overline{E}_j^T > 0$, such that $\| H_{\omega y} \|_2^2 = tr\,[D^T(\rho)D(\rho) + B^T(\rho)RB(\rho)]$, and $E^T(\rho)RE(\rho) - A^T(\rho)RA(\rho) - C^T(\rho)C(\rho) > 0$.

- *Sufficiency*: Assuming that the condition (53) is satisfied, then multiplying by the left by $\begin{bmatrix}\mathbb{I} & -A_j^T & C_j^T\end{bmatrix}$ and the right by $\begin{bmatrix}\mathbb{I} & -A_j^T & C_j^T\end{bmatrix}^T$, the following inequality is obtained:

$$E_j^T Q E_j - A_j^T R A_j - C_j^T C_j > 0, \qquad (62)$$

which represents the corresponding LMI extended characterization of (51), that is exactly the necessary and sufficient condition for descriptor LPV system (47) to be admissible with a sub-optimal performance in $\mathcal{H}_2$.

- *Necessity*: Assuming that the condition (62) is satisfied, then there exist a matrix G such that $G + G^T - R > 0$, therefore

$$\begin{bmatrix} E_j^T Q E_j - A_j^T R A_j - C_j^T C_j & 0 \\ 0 & G + G^T - R \end{bmatrix} > 0, \qquad (63)$$

which is equivalent to

$$\begin{bmatrix} E_j^T Q E_j - A_j^T R A_j & 0 & C_j^T \\ 0 & G + G^T - R & 0 \\ C_j & 0 & \mathbb{I} \end{bmatrix} > 0. \qquad (64)$$

In effect, (63) is obtained pre-multiplying (64) by $\mathfrak{T} = \begin{bmatrix} \mathbb{I} & 0 & -C_j^T \\ 0 & \mathbb{I} & 0 \end{bmatrix}$ and post-multiplying it by $\mathfrak{T}^T$. Consequently, in order to obtain (53), the LMI (64) is multiplied by the right by $\mathfrak{U} = \begin{bmatrix} \mathbb{I} & 0 & 0 \\ A_j & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{bmatrix}$ and by the left by $\mathfrak{U}^T$. Thus, by choosing $= A_j^T R - A_j^T G^T$, the condition (53) is obtained. Finally, *iii)* is the dual equivalent of *ii)*.

Remark 2. *A similar procedure is applied for the characterization of the $\mathcal{H}_\infty$-norm, according to the BRL (see Section 3), as a strict LMI.*

4.1 $\mathcal{H}_2$-Control for Descriptor LVP System

From of the LMI characterization of the $\mathcal{H}_2$-norm, we are interesting in to design a control law by state feedback in order that the closed-loop system be robustly admissible.

Lemma 4.3. *The discrete-time descriptor LPV system (47) is admissible and satisfies $\| H_{\omega y} \|_2^2 < \mu$ via a state feedback controller (33), if and only if: there exists a positive definite $Q = Q^T \in \mathbb{R}^{n\times n}$, a symmetric matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ and the matrices $W \in \mathbb{R}^{p\times p}$ with $tr\,(W) < \mu$, G and Y satisfying, $\forall j = 1,2, \dots N$, the following LMI*

$$\begin{bmatrix} E_j Q E_j^T & A_j G + B_j Y & B_{\omega j} \\ G^T A_j^T + Y^T B_j^T & G + G^T - Q + \overline{\overline{E}}_j S \overline{E}_j^T & 0 \\ B_{\omega j}^T & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (65)$$

$$\begin{bmatrix} W & C_j G & D \\ G^T C_j^T & G + G^T - Q + \overline{\overline{E}}_j S \overline{E}_j^T & 0 \\ D_j^T & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (66)$$

where the feedback gain is given by

$$\boldsymbol{K} = Y G^{-1} \quad (67)$$

Proof

Applying item iv) of Theorem 4.1, using the dynamic matrix of the closed-loop $A_j + B_j\boldsymbol{K}$, then the change of variable $Y = \boldsymbol{K}G$ must be used.

Remark 4.2. *When the robust admissibilization is considered with finite poles in a D-stability region of the complex Z-plane (see Section 2.3), which is defined by the parameters $\sigma, r \in \mathbb{R}$ and $y \leq b < r$, $\sigma + r < 1$, then the dynamic matrix corresponds to the matrix $\frac{A+BK-\sigma\mathbb{I}_n}{r}$, consequently the LMI (65) is transformed to*

$$\begin{bmatrix} \left(r^2 - y^2\left(\frac{r^2}{b^2} - 1\right)\right) E_j Q E_j^T & \Lambda & r B_{\omega j} \\ \Lambda^T & \Phi & 0 \\ r B_{\omega j}^T & 0 & \mathbb{I} \end{bmatrix} > 0 \quad (68)$$

with $\Phi = G + G^T - Q + \overline{\overline{E}}_j S \overline{E}_j^T$ and $\Lambda = A_j G + B_j Y - \sigma G$.

4.2 Example

Let consider the descriptor polytopic system

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}$$

$$A = \begin{bmatrix} -.25 & 0.1\rho & 0.01\rho \\ -0.5 & 0.5 - 0.01\rho & 2 - 0.01\rho \\ 0.75 & -1 + 0.005\rho & -1.5 + 0.005\rho \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 0 & 2 \end{bmatrix}, \quad D_\omega = \begin{bmatrix} 0.01 & -0.5 \end{bmatrix}$$

The matrix A is assumed to be uncertain with $\rho \in [-1,1]$. Considering $\overline{E}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\mu = 1$, then

$$G = 1.0e+04 \times \begin{bmatrix} 0.0665 & -0.1966 & 0.0196 \\ -0.0215 & 2.6149 & -0.0070 \\ 0.0221 & -0.7053 & 0.0072 \end{bmatrix}$$

$$Y = 1.0e+04 \times [-0.0759 \quad 1.8041 \quad -0.0221],$$
$$S = 7.9698e+08$$

Therefore, the feedback gain is

$$\boldsymbol{K} = [-1.3242 \quad 0.9999 \quad 1.5183]$$

In order to evaluate the robust admissibility, Figure 8 and 9 shows the closed-loop pole distribution with respect to the parametric variation and its projection in the complex plane. There is evident that the closed loop system is robustly admissible. Analogous, Figure 10 shows the variation of the $\mathcal{H}_2$-norm for the closed-loop system with respect to ρ, which allows to conclude that $\| H_{\omega y} \|_2^2 < \mu$ is satisfied.

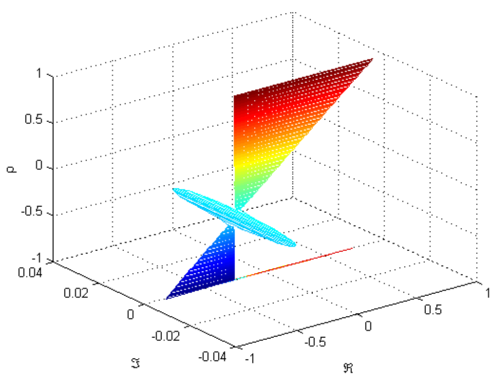

Fig. 8. The pole distribution of the closed-loop system. Distribución de polos del Sistema en lazo cerrado.

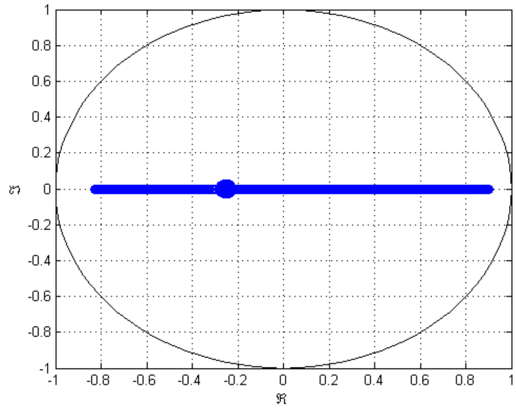

Fig. 9. The pole projection for the closed-loop system. Proyección de polos del Sistema en lazo cerrado.

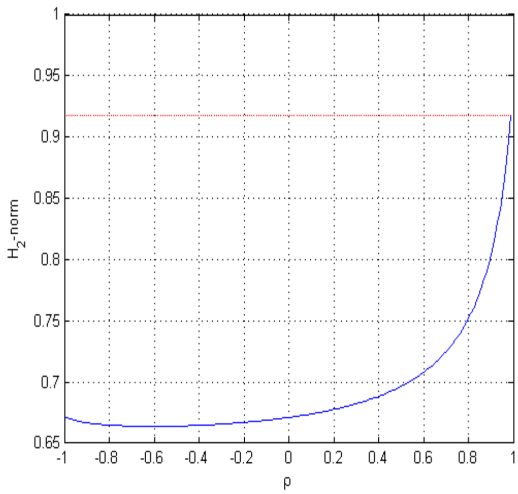

Fig. 10. $\mathcal{H}_2$-norm for closed-loop system $H_{\omega y}(z)$. Norma-$\mathcal{H}_2$ para el sistema en lazo cerrado $H_{\omega y}(z)$.

5 Conclusions

A method for the robust admisibilization of discrete-time LPV-type descriptor systems has been presented. The technique is based on the synthesis of robust controllers from satisfying performance index in $\mathcal{H}_2$-$\mathcal{H}_\infty$. In order to avoid computational difficulties, an extended characterization of the $\mathcal{H}_2$–$\mathcal{H}_\infty$norms has been presented, which allows obtaining strict and less conservative linear matrix inequalities. Thus, the necessary and sufficient admissibility conditions in strict linear matrix inequality have also been derived. The results obtained can be extended to robust admissibility by considering the location of poles in LMI regions.

References

Barbosa, Karina A., Carlos E. de Souza, and Daniel Coutinho. (2018). "Admissibility Analysis of Discrete Linear Time-Varying Descriptor Systems." *Automatica* 91: 136–43.

Briat, Corentin. (2008). "Robust Control and Observation of LPV Time-Delay Systems." PhD thesis, Institut National Polytechnique de Grenoble-INPG.

Chaabane, M, F Tadeo, D Mehdi, and M Souissi. (2011). "Robust Admissibilization of Descriptor Systems by Static Output-Feedback: An LMI Approach." *Mathematical Problems in Engineering* 2011 (1): 1–10.

Chadli, Mohammed, and Mohamed Darouach. (2012). "Novel Bounded Real Lemma for Discrete-Time Descriptor Systems: Application to $\boldsymbol{\mathcal{H}_\infty}$ Control Design." *Automatica* 48 (2): 449–53.

Chadli, Mohammed, and Mohamed Darouach. (2013). "Further Enhancement on Robust $\boldsymbol{\mathcal{H}_\infty}$ Control Design for Discrete-Time Singular Systems." *IEEE Transactions on Automatic Control* 59 (2): 494–99.

Chadli, M, Peng Shi, Z Feng, and J Lam. (2017). "New Bounded Real Lemma Formulation and Control for Continuous-Time Descriptor Systems." *Asian Journal of Control* 19 (6): 2192–98.

Chang, Xiao-Heng, and Jian Wang. (2021). "$\boldsymbol{l_2 - l_\infty}$ Control for Discrete-Time Descriptor Systems." *IEEE Access* 9: 144017–24.

Duan, Guang Ren. (2010). *Analysis and Design of Descriptor Linear Systems*. New York: Springer.

Duan, Guang-Ren, and Hai-Hau Yu. (2013). *LMIs in Control Systems: Analysis, Design and Applications. Center for Control Theory*; Guidance Technology, China: CRC Press.

Feng, Yu, and Mohamed Yagoubi. (2017). *Robust Control of Linear Descriptor Systems*. Vol. 194. Springer.

Fujimori, Atsushi. (2004). "Descriptor Polytopic Model of Aircraft and Gain Scheduling State Feedback Control." *Trans. of the Japan Society for Aeronautical and Space Sciences* 47 (156): 138–45.

González, Antonio, Víctor Estrada-Manzo, and Thierry-Marie Guerra. (2017). "Gain-Scheduled $\boldsymbol{\mathcal{H}_\infty}$ Admissibilisation of LPV Discrete-Time Systems with LPV Singular Descriptor." *International Journal of Systems Science* 48 (15): 3215–24.

Grman, Ĺubomír, Danica RosinovÁ, Vojtech Veselỳ, and Alena KozÁ KovÁ. (2005). "Robust Stability Conditions for Polytopic Systems." *International Journal of Systems Science* 36 (15): 961–73.

Hilhorst, G., G. Pipeleers, R. C. L. F Oliveira, P. L. D. Peres, and J. Swevers. (2014). "On Extended LMI Conditions for $\boldsymbol{\mathcal{H}_2}$-$\boldsymbol{\mathcal{H}_\infty}$ Control of Discrete-Time Linear Systems." In Proc. 19th IFAC World Congress, 1:9307–12. Cape Town, South Africa: IFAC.

Ikeda, Masao, Tickwoon LEE, and Eiho Uezat. (2000). "A Strict LMI Condition for $\boldsymbol{\mathcal{H}_2}$ Control of

Descriptor Systems." In Proc. 39^{th} IEEE Conference on Decision and Control, 1:601–4. IEEE.

Krokavec, Dušan, and Anna Filasová. (2019). "A New D-Stability Area for Linear Discrete-Time Systems." *Archives of Control Sciences* 29.

Luenberger, D. G. (1977). "Dynamic Equations in Descriptor Form." *IEEE Trans. Automat. Control* 22 (3): 312–21.

Mao, W-J. (2012). "Robust Stability and Stabilisation of Discrete-Time Descriptor Systems with Uncertainties in the Difference Matrix." *IET Control Theory & Applications* 6 (17): 2676–85.

Peaucelle, Dimitri, Denis Arzelier, Olivier Bachelier, and Jacques Bernussou. (2000). "A New Robust D-Stability Condition for Real Convex Polytopic Uncertainty." *Systems & Control Letters* 40 (1): 21–30.

Pipeleersa, Goele, Bram Demeulenaere, Jan Sweversa, and Lieven Vandenbergheb. (2009). "Extended LMI Characterizations for Stability and Performance of Linear Systems." *Systems & Control Letters* 58 (7): 510–18.

Rodríguez, Carlos, Karina A Barbosa, and Daniel Coutinho. (2018). "Robust $\mathcal{H}_\infty$ State-Feedback Design for Discrete-Time Descriptor Systems." IFAC-PapersOnLine 51 (25): 78–83.

Scholz, Lena. (2015). "*Control Theory of Descriptor Systems: Lecture Notes*." TU Berlin (WS2014/15).

Shamma, Jeff S. (2012). "Control of Linear Parameter Varying Systems with Applications." In, edited by Javad Mohammadpour and Carsten W. Scherer, 3–26. Springer.

Sjöberg, Johan. (2005). *Descriptor Systems and Control Theory*. Linköping University Electronic Press.

Zhang, Baoyong, Shengyuan Xu, Qian Ma, and Zhengqiang Zhang. (2019). "Output-Feedback Stabilization of Singular LPV Systems Subject to Inexact Scheduling Parameters." *Automatica* 104: 1–7.

Zhang, Gaomin, Yuanqing Xia, and Peng Shi. (2008). "New Bounded Real Lemma for Discrete-Time Singular Systems." *Automatica* 44 (3): 886–90.

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