Kink-soliton explosions

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Abstract

The dynamics of kink-solitons in generalized Klein-Gordon equations for excitable media and spatial perturbations are investigated. A mechanism for kink-soliton explosion is presented, along with the analytically obtained conditions for the phenomena to happen. Computer simulations are used to visualize and corroborate the analytical findings. These findings can explain some of the phenomena that recently have been reported to occur in excitable media.

Resumen

Investigamos la dinámica de los kink-solitones de una ecuación generalizada de seno-Gordon, para un medio excitable y perturbado por fuerzas espaciales. Presentamos un mecanismo para la explosión del kink-solitón, además de una condición analítica que permite determinar si la explosión aparece. Simulaciones numéricas corroboran nuestros resultados analíticos. Estos resultados ayudan a explicar algunos de los fenómenos recientemente reportados en medios excitables.

Excitable media are physical, chemical or biological systems in which besides energy dissipation, in some disturbed regions, there is energy supply (Zykov, 1988; Holden et al., 1991). These media can support wave motion without attenuation. Topological defects breakup have been observed in this systems (Barkley, 1992).

Kink-solitons are the simplest examples of a general phenomenon called topological defects. This set of phenomena also includes: vortices and spirals (Cross and Hohenberg, 1993; Aranson and Kramer, 2002; Mello et al., 1998). Although these objects possess different origin and nature in different physical systems, they possess very similar dynamical properties (Cross and Hohenberg, 1993; Aranson and Kramer, 2002; Mello et al., 1998).

Topological kink-solitons possess important applications in condensed matter physics (Kivshar and Malomed, 1989). Therefore, it is very important that we understand how the breakup/explosion happens.

In the present communication we investigate excitable media systems described by the following generalized Klein-Gordon equations:

$$\varphi_{tt} + R(\varphi_t) - \varphi_{xx} - G(\varphi) = F(x), \qquad (1)$$

where $G(\varphi) = -\mathrm{d}U(\varphi)/\mathrm{d}\varphi$, $U(\varphi)$is a potential function with at least two minima φ_1, φ_3 and a maximum φ_2, such that $U(\varphi_1) = U(\varphi_3) = 0$, $R(\varphi_t)$ are nonlinear dissipative terms, and $F(x)$ represents external perturbations. We are interested in topological kink-solitons between the points φ_1 and φ_3. The famous sine-Gordon and φ^4-systems are particular cases of Eq. (1).

We will present a mechanism for soliton explosions; and we will show that in some cases, while some conditions hold, the soliton explosion is permanent.

In excitable media self-sustained dynamical patterns are possible. In this respect, all the media and physical systems with nonlinear damping, where self-sustained oscillations can exist, are very similar.

Kink-soliton bearing excitable media systems as the following:

$$\varphi_{tt} + R(\varphi_t) - \varphi_{xx} - G(\varphi) = 0, \tag{2}$$

where $\mathrm{d}R(\varphi_t)/\mathrm{d}\varphi_t$ is negative for small values of $|\varphi_t|$ and positive elsewhere, can support kinks moving with a constant velocity. An example of this kind of systems can be realized in practice using a Josephson junction transmission line where the resistor is a negative-resistance twin-tunnel-diode circuit or a twin-transistor system (Chua et al., 1987). In this case, $R(\varphi_t) = -b\varphi_t + a\varphi_t^3$ is a good model.

From Eq. (2) we obtain that kinks that move without changes of shape and velocity correspond to solutions of the equation

$$\varphi_{zz} - R(-w\varphi_z) + G(\varphi) = 0, \tag{3}$$

where $z = (x - vt)/\sqrt{1-v^2}$ and $w = v/\sqrt{1-v^2}$, v being a velocity of the kink that satisfies the equation

$$\int_{-\infty}^{\infty} R(-w\varphi_z)\varphi_z dz = 0. \tag{4}$$

Eq. (4) is satisfied by three values of the velocity v: $v = 0$, $v = v_1 > 0$, and $v = v_2 = -v_1 < 0$. The velocity $v = 0$ is unstable.

However, the solution is a kink-soliton only if function

$$V(\varphi) = U(\varphi) + \int_{-\infty}^{\infty} R\left(-w\sqrt{2U(\varphi)}\right)\mathrm{d}\varphi \tag{5}$$

satisfies the condition $V(\varphi) > 0$ in the whole interval $\varphi_1 < \varphi < \varphi_3$. Suppose $R(\varphi_t)$ possesses two local extrema: a maximum and a minimum such that the value of $|R(\varphi_t)|$ at these extrema is R_m. If this value is comparable with the absolute value of the extrema of $G(\varphi)$ (let us call it G_m) in the interval $\varphi_1 < \varphi < \varphi_3$, then the condition $V(\varphi) > 0$ may be not satisfied. In fact, if $R_m > G_m$ this condition is certainly not satisfied. When this happens, the kink becomes a highly nonstationary state.

If $R(\varphi_t) = -b\varphi_t + a\varphi_t^3$, then $R_m = \frac{2}{3} b\sqrt{b/3a}$. So, for a given $G(\varphi)$, and a fixed a, parameter b is the key. For small b, the kink can move smoothly with a constant velocity. For larger values of b, a big transformation will occur. This phenomenon can be observed in Figures 1(a) and 1(b).

Another way to experiment negative damping is when the damping coefficient is a function of x:

$$\varphi_{tt} + \Gamma(x)\varphi_t - \varphi_{xx} - G(\varphi) = F(x). \quad (6)$$

Where $F(x)$ has a stable zero, say $x = x_0$, the center of mass of a kink can perform damped oscillations around x_0; and $\Gamma(x)$ is negative in a neighborhood of x_0 and positive elsewhere. This can be done in a chain of nonlinear oscillators using negative-resistance circuits (Chua et al., 1987) only in some small interval of the chain.

An example of $\Gamma(x)$ with the required features is $\Gamma(x) = \gamma\left[1 - L\,\mathrm{sech}^2(Dx)\right]$, where $(1 - L) < 0$.

If we are not careful, the kink can explode also in this system. In fact, if $\Gamma(0) > G_m$, then we can observe a very turbulent behavior as that shown in Fig. 2.

The situations discussed in here, which lead to very complex spatiotemporal behaviors, start with soliton breakups after a Hopf bifurcation (see figures 1b and 2). These have already been documented in experiments (Barkley, 1992). We believe that this result shows that very similar phenomena can occur with kink-solitons in Klein-Gordon systems. We have been able to produce defect-mediated turbulence after a Hopf bifurcation generated by nonlinear damping.

In Eq. (2) (with $R(\varphi_t) = -b\varphi_t + a\varphi_t^3$) the energy supply is described by the term $-b\varphi_t$. If coefficient b is very small, the defects are stable. However, if b is larger than a critical value, the kink-soliton breakup occurs. Our guess is that, in general in excitable systems, the physical elements that are responsible for the energy supply may also be responsible for some of the defect breakups.

Many of the known results about excitable systems have been obtained by real and numerical experiments. Here we have presented an analytical theory of the breakup of Klein-Gordon solitons. We believe that our results can enlighten some of the still obscure phenomena that occur in excitable systems.

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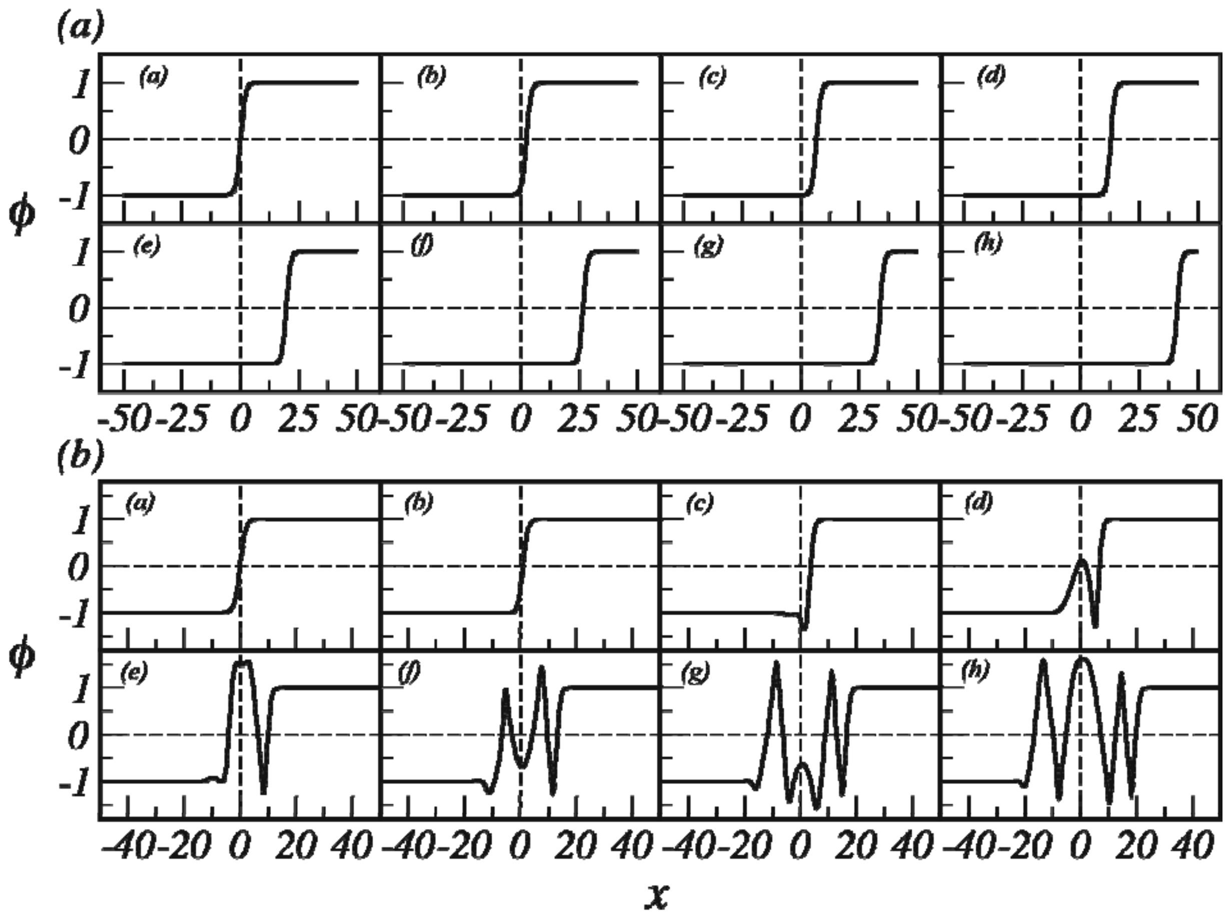

Figure 1: (a) Despite dissipation, Eq. (2) can sustain a kink with constant velocity due to negative resistance. Here $G(\varphi)=(\varphi-\varphi^3)/2$, $R(\varphi_t)=-b\varphi_t + a\varphi_t^3$, $a = 1$, $b = 0.05$. (b) Kink-soliton breakup due to nonlinear damping. If $R(\varphi_t)=-b\varphi_t + a\varphi_t^3$, and parameter b is larger that some critical value, the kink will explode. The parameters take the same values as in Fig. 1(a) but $b = 0.7$.

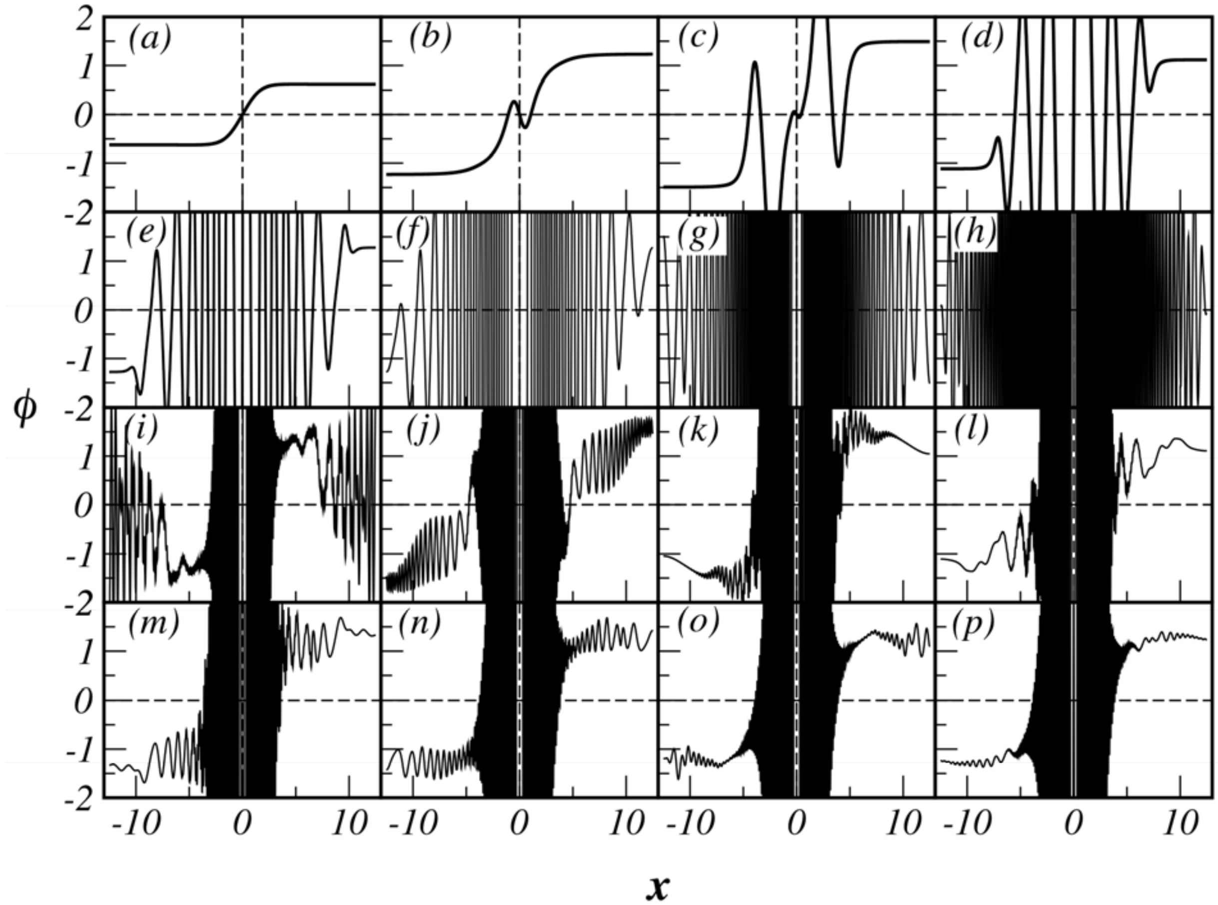

Figure 2: Highly nonstationary spatiotemporal dynamics produced by Eq. (6) if $\Gamma(0) > G_m$. Here $\Gamma(x) = 1 - l\,\mathrm{sech}^2(Bx)$, $G(\varphi)=(\varphi - \varphi^3)/2$, $F(x)= A\tanh(Bx)$, $l = 6$, $D = 0.65$, $A = 0.45$, $B = 0.65$.